Staying cool and getting cooler

Modeling the thickness of sea ice is very complicated if you take into account the brine channels and salt crystals and snow cover. However, a few simplifying assumptions allow us to develop a reasonable model. We will assume, among other things,

1. no snow cover and a known temperature T_{top} at the top of the sheet of sea ice,
2. a constant temperature gradient vertically through the slab of ice, and
3. no other funny business (thermal inertia, heat flux, internal heat sources, etc.).

The first assumption is that the temperature T at the top of the ice slab is known every day. Call this temperature $T_{top}(t)$, depending on time t in days. [1]

1. The second assumption is that the temperature gradient through the slab of ice is constant. Let z be the depth in centimeters from the top of the ice sheet to the bottom. How can you express the second assumption in mathematical symbols? (Don't think too hard!)

   Solution: Write it as $\frac{\partial T}{\partial z} = const$.

2. Label the temperature at the bottom of the sheet of sea ice by T_f. Then if the thickness of the sea ice is $H(t)$, the top temperature is $T_{top}(t)$, and the change in temperature with respect to z is constant, write an expression for $\frac{\partial T}{\partial z}$ in terms of T_f, $T_{top}(t)$, and $H(t)$.

   Solution: Write it as $\frac{\partial T}{\partial z} = \frac{T_f - T_{top}(t)}{H}$.

3. We can estimate that

$$\frac{dH}{dt} = 5.445 \frac{\partial T}{\partial z}.$$

   Use your result from above to rewrite this as a differential equation, replacing $\frac{\partial T}{\partial z}$.

   Solution: $\frac{dH}{dt} = 5.445 \frac{T_f - T_{top}(t)}{H}$.

[1] Heavily based on Leppäranta, Matti(1993), 'A review of analytical models of sea-ice growth', Atmosphere-Ocean, 31: 1, 123 - 138.

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4. Use separation of variables to write an expression that will let you solve this differential equation. Use H_0 to indicate the thickness of the ice at time zero, and let τ be a dummy variable since there are too many ts. Fill in the blanks:

$$\int_{H_0}^{H(t)} \underline{\hspace{2em}} dH = \int_{\underline{\hspace{1em}}}^{t} \underline{\hspace{2em}} d\tau$$

Solution: Rewrite as $H(t)dH = 5.445(T_f - T_{top}(t))dt$. Yes, I know this is in some sense illegal, but I like it. Now, since both sides depend on time t, use dummy variables H and τ to fill in:

$$\int_{H_0}^{H(t)} H dH = \int_0^t 5.445(T_f - T_{top}(\tau))d\tau.$$

5. Solve the equation above. Be very careful: there is some trickiness here! What do you know how to integrate and what can't you integrate?

Solution: Integrate to get

$$\frac{H(t)^2 - H_0^2}{2} = 5.445 \int_0^t (T_f - T_{top}(\tau))d\tau.$$

Since we don't know how T_{top} depends on time, we can't integrate this.

6. Solve for $H(t)$ to get an equation for the thickness of sea ice given an initial thickness of H_0.

Solution: If we write $S(t) = \int_0^t (T_f - T_{top}(\tau))d\tau$, we can get

$$H(t) = \sqrt{10.89S(t) - H_0^2}.$$

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7. Write $S(t) = \int_0^t (T_f - T_{top}(\tau))d\tau$. This is called the sum of negative degree-days. Start with no ice and assume that $S(t)$ grows linearly with respect to time t. What will the graph of $H(t)$ look like?

   Solution: It will grow like a constant multiple of $\sqrt{t}$.

8. If there is no ice to start with but temperatures are ten degrees below the freezing point for 100 days, what will the thickness of the sea ice be by day 100?

   Solution: No ice to start with means $H_0 = 0$, Temperature ten below freezing for 100 days means we're taking $S(100) = \int_0^{100}(0 - 10)d\tau$, which is just 1000. Then $H(100)$ is about 104 centimeters, or about a meter.

9. Barrow, Alaska, averages 160 days below freezing a year. If the average temperature during these days is four degrees below freezing, how thick might nearby sea ice get? If the average temperature during these days is ten degrees below freezing, how thick might nearby sea ice get?

   Solution: I get approximately 83 cm and 132 cm. Temperature makes a difference!

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10. The *equilibrium thickness* of sea ice that forms year after year can be estimated using the equation above. Let's call the equilibrium thickness of the ice H_{eq}, and let's assume that at the end of the ice growth season it melts ΔH centimeters. During the next ice growth season, it will then grow from $H_{eq} - \Delta H$ to H_{eq} again. Solve for H_{eq} using the equation you derived above.

Solution: Solve

$$H_{eq}^2 = 10.89S(t) + (H_{eq} - \Delta H)^2$$

to get

$$H_{eq} = \frac{10.89S(t)}{2\Delta H} + \frac{\Delta H}{2}.$$

11. If $\sqrt{10.89S(t)} = 200$ centimeters, and the ice thins by 50 centimeters each warm season, what will H_{eq} be?

Solution: Get $425 = H_{eq}$. This is actually not far off from observed values of 3 to 4 meters thickness for sea ice in the Arctic.