Leaky Planet

Earth has a gas leak: a steady stream of air molecules quietly hiss away from our outer atmosphere. What threat does this pose to human existence?

First, escape velocity. Derive how fast something must be going to escape Earth's gravitational field. This will require us to solve a differential equation and look at some limits as time goes to infinity!

1. Fill in the blanks: Acceleration $a(t)$ is the derivative with respect to time of ______, and thus can be written $\frac{d}{dt}$___.

2. Fill in the blanks: Think about two objects distance $r(t)$ apart. If you like, pretend one is staying still and the other is moving toward or away from the first. The velocity $v(t)$ of the moving object is the derivative with respect to time of ______, and thus can be written $\frac{d}{dt}$___.

3. Newton's Second Law says that $F = ma(t)$: the force acting on an object with mass m equals m times the acceleration $a(t)$ of the object. At the same time, Newton's law of universal gravitation says that two objects with masses M and m exert a gravitational force on each other that depends on the distance $r(t)$ between them and a constant G:

$$F = G\frac{mM}{r^2}.$$

   When these forces balance each other, they point in opposite directions and we have

$$ma = -G\frac{mM}{r^2}.$$

   Cancel anything you can, and rewrite the left-hand side using the result of question (1).

$$\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_$$

4. Apply the Chain Rule to split $\frac{dv}{dt}$ into the product of two derivatives. (Is this legitimate?)

5. Rewrite the term in the Chain Rule product containing dt using question (2).

$$\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_$$

   This gives a *differential equation*, relating the derivative of $v(r)$ to r.

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6. Differential equations and integrals are closely related. (That dx at the end of an integral is there for a reason!) We can rewrite your differential equation from page 1 as follows, almost as if we're multiplying through by dr and then integrating. Fill in the blanks:

$$\int_{v_0}^{v(t)} \underline{\hspace{2cm}} dv = \int_{r_0}^{r(t)} \underline{\hspace{2cm}} dr.$$

   Here r_0 and v_0 are initial distance apart and initial velocity, respectively, while $r(t)$ and $v(t)$ are distance apart and velocity after time.

7. Integrate! Evaluate at the endpoints! You should get an equation in terms of v_0, r_0, $v(t)$, $r(t)$, and G and M.

8. Now we need to let $t \to \infty$. Since the two objects are moving apart from each other,

$$\lim_{t\to\infty} r(t) = \underline{\hspace{2cm}}$$

   and

$$\lim_{t\to\infty} v(t) = 0.$$

9. Use these values to evaluate your expression from question (7) as $t \to \infty$. Use good limit notation.

10. Solve for v_0. This is the escape velocity for any object of mass M that starts at distance r_0 from, say, the center of the earth.

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11. **Now, gas escape.** Escape velocity for an object at a distance r_0 from the center of the earth is
$$v = \sqrt{\frac{2GM}{r_0}},$$
where $G = 6.6738 \times 10^{-11}\ m^3/(kg \cdot s^2)$. What is the escape velocity of an object $9,000$ km above the earth? Use the fact that the mass of the earth is $M = 5.9736 \times 10^{24}\ kg$ and the radius of the earth is about $6,400$ km.

12. The root mean square velocity of the molecules in a gas is
$$v_{rms} = \sqrt{3RT}M_m,$$
where R is the universal gas constant and M_m is the molar mass of the gas in kilograms per mole. For the hydrogen molecule, this equation simplifies to
$$v_{rms} = \sqrt{.7274T}.$$
How hot must hydrogen at 9000 kilometers be to escape the earth's pull?

13. $9,000$ km above the earth's surface we're at the very edge of the earth's exosphere. There is so little air there that temperatures are within a few degrees of zero Kelvin at night and up to 1,000 degrees Kelvin when heated by the sun. According to your predictions, should hydrogen gas be leaking from the exosphere into space?

14. Repeat the two calculations above for 18 km above the earth's surface, the top of our troposphere. How hot must hydrogen molecules be to escape gravity's pull from 18 km? Are we that hot?