Head in the clouds: how'd we get up here?

The derivation for the equation for atmospheric pressure P at a given altitude z is surprisingly straightforward. This worksheet will guide you to discovering a simple model for pressure for yourself from first principles, with just one major assumption (that does turn out to be wrong). What else could you prove?

Equilibrium: setting up the equations

We are roughly living in a hydrostatic equilibrium. That means that the change in pressure over an infinitesimal unit of altitude equals the opposite of the gravitational force on that infinitesimal piece of air: overall, our atmosphere isn't all going away or all being sucked into the surface of the earth by gravity.

1. The ideal gas law states that

$$P = \rho RT, \tag{1}$$

where ρ is the density of air, T is temperature, and $R = 287.053$ Joules per kilogram Kelvin is the gas constant for air. If temperature T increases while ρ stays the same, does P increase or decrease?

Solution: Since P and T are proportional, if T increases then P increases. Point out to students that Kelvin is a scale for temperature starting from absolute zero and going up – you can't have negative degrees Kelvin.

2. The force of gravity on air can be deduced from the familiar equation $F = ma$, where F is force, m is mass, and a is acceleration. F is measured in Newtons and m in kilograms. The units, then, are:

$$\text{Newtons} = \text{kilograms} \cdot \text{meters/seconds}^2$$

I claim that

$$\frac{dP}{dz} = -\rho \cdot g. \tag{2}$$

If you know that pressure is measured in Pascals (newtons/meter^2), fill in the following blanks with the appropriate units to check that at least the units work out!

$$\frac{\Delta N/m^2}{\Delta \underline{z}} = \frac{kg}{m^3} \cdot \underline{\frac{m}{s^2}}$$

Simplify, using the conversion for Newtons above, and confirm:

$$\underline{\frac{kg}{m^2 \cdot s^2}} = \underline{\frac{kg}{m^2 \cdot s^2}}$$

Solution: This is called dimensional analysis, and it is very useful for both error-checking and hypothesis generation.

Copyright EarthCalculus 2013

Head in the clouds: how'd we get up here?

3. Solve the equation (1) for ρ, and substitute it into equation (2). Write the result below.

> **Solution:**
> $$\frac{dP}{dz} = -\frac{g}{RT}P$$

Differential equation and assumptions

4. Now you've got what's called a *differential equation*, relating P and $\frac{dP}{dz}$. We can assume for altitudes less than 11 kilometers that g and R are constant. If you assume temperature T is also constant, you have $\frac{dP}{dz} = constP$. Guess the function $P_{hyp}(z)$ that satisfies this equation.

> **Solution:** Guess that $P_{hyp}(z) = P_0 e^{const \cdot z}$, where P_0 is the pressure at sea level. It works – check by differentiating! So we get $P_{hyp}(z) = P_0 e^{-\frac{g}{RT}z}$. This is called the hypsometric equation. It is accurate at low altitudes.

5. What is the inverse of this function $P_{hyp}(z)$?

> **Solution:**
> $$\ln \frac{P_{hyp}(z)}{P_0} = -\frac{g}{RT}z,$$
> so rewriting,
> $$z = -\frac{RT}{g} \ln \frac{P_{hyp}(z)}{P_0}.$$

6. (Challenge points) Assuming that temperature is constant is actually fairly terrible – we all know that high altitudes are much colder. A better assumption is that temperature is a function $T(z) = T_0 - 0.0065z$, with T_0 temperature at sea level. Plug that into your differential equation from question (3). Can you guess a function $P(z)$ that satisfies your new differential equation?

Copyright EarthCalculus 2013